AP Calculus BC

Unit 7 – Applications of Integration

AP Calculus BC – Worksheet 49 **Estimating with Areas – Riemann Sums**

1

Find a Right Rectangular Approximation for the area under $f(x) = -x^2$ on the interval $[0, 2]$ using four subintervals. Explain why this approximation is either an underestimate or overestimate.

2

Find a Left Rectangular Approximation using four subintervals for $f(x) = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$. Explain with this approximation is either an underestimate or overestimate.

3

Refer to the region *R* enclosed between the graph of the function $f(x) = 2x - x^2$ and the *x*-axis for $0 \le x \le 2$

a) Approximate the area of the region using $RRAM_4$.
b) Approximate the area of the region using $LRAM_4$.
c) Approximate the area of the region using $MRAM_2$.
d) Approximate the area of the region using $TRAP_2$.

4

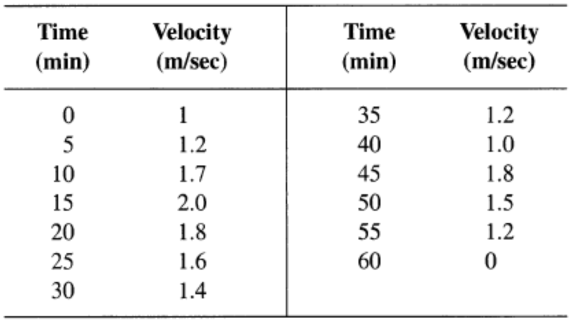
Consider the continuous function $f(x)$ such that $f(x) > 0$ for $[0,1]$. Selected values of $f(x)$ are given in the table below. Use the table of values to approximate the area under $f(x)$ using the Riemann Sum indicated.

x	0	0.25	0.5	0.75	1.0
$f(x)$	1.0	0.8	1.3	1.1	1.6

a) Left Approximation using 4 subintervals.
b) Trapezoidal Approximation using 4 subintervals.
c) Midpoint Rectangular Approximation using 2 subintervals

5

You are sitting on the bank of a river watching the incoming tide carry a bottle upstream. Find an estimate for $\int_0^{60} v(t)\,dt$ using 6 subintervals of length 10. Explain the meaning of your answer in the context of the problem

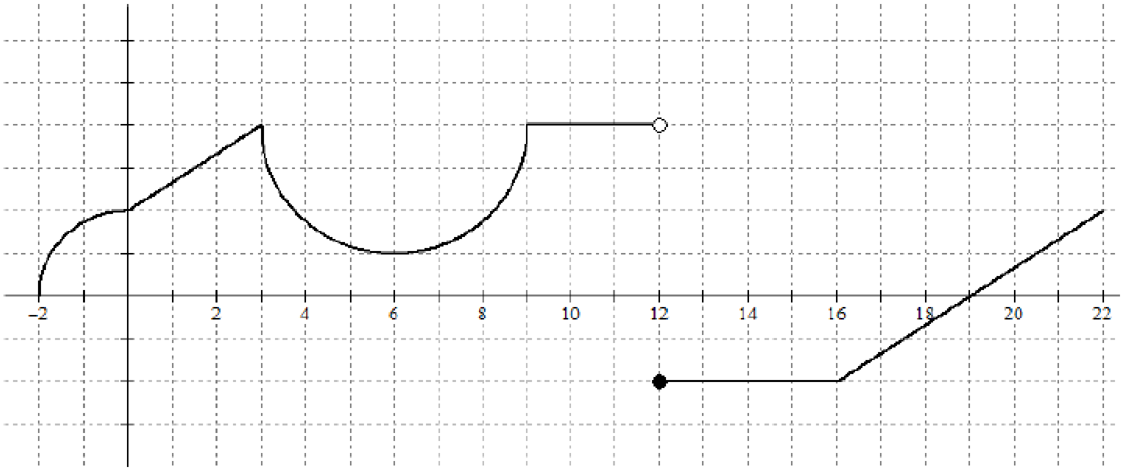


Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

AP Calculus BC – Worksheet 50 **Definite Integrals as Riemann Sums**

In the chart below, either a definite integral or a limit of a Riemann sum has been provided. Fill in the box with the corresponding missing information

	Definite Integral	**Limit of Riemann Sum**
1	$\int_0^6 \sqrt{2x+1}dx$	
2		$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+\frac{5i}{n}\right)^2-3\right]\cdot\frac{5}{n}$
3		$\lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(1+\frac{4i}{n}\right)-7\right]\cdot\frac{4}{n}$
4	$\int_{-2}^4 x^3dx$	
5		$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{\left(-2+\frac{2i}{n}\right)^2+1}\right]\cdot\frac{2}{n}$
6	$\int_0^3 e^xdx$	
7		$\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{\pi}{2}+\frac{\pi i}{n}\right)\cdot\frac{\pi}{2n}$



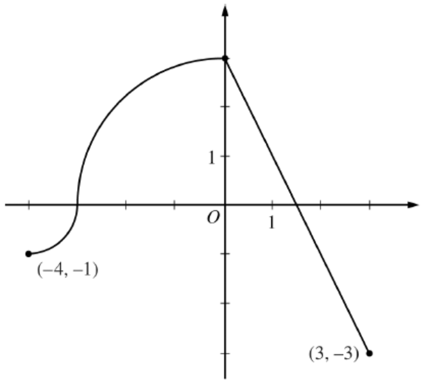
The graph above consists of a quarter circle, a half circle, and four line segments. For each of the expressions below (either definite integral or limit of Riemann sum), fill in the missing information. Then determine the value of each integral using geometric formulas (without using a calculator).

	Limit of Riemann Sum	**Definite Integral**	**Value of Definite Integral**
8	$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(-2)\left(\frac{4}{n}\right)$		
9		$\int_{3}^{9}\left[4-\sqrt{9-(x-6)^{2}}\right]dx$	
10		$\int_{0}^{3}\left(\frac{2}{3}x+2\right)dx$	
11	$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left[\frac{2}{3}\left(\left(16+\frac{6k}{n}\right)-19\right)\right]\cdot\frac{6}{n}$		
12		$\int_{9}^{12}4dx$	

AP Calculus BC - Worksheet 51 **The Accumulation Function and FTC**

In exercises 1-8, find the derivative.

1. $y=\int_0^{x^2} e^{t^2}\,dt$	2. $f(x)=\int_6^{x^2} \cot 3t\,dt$	3. $F(x)=\int_x^7 \sqrt{2t^4+t+1}\,dt$	4. $y=\int_{-\pi}^{x} \frac{2-\sin t}{3+\cos t}\,dt$
5. $y=\int_{\sqrt{x}}^{0} \sin(r^2)\,dr$	6. $f(x)=\int_{x^2}^{x^3} \cos(2t)\,dt$	7. $g(x)=\int_7^x \frac{1+t}{1+t^2}\,dt$	8. $g(x)=\int_x^6 \ln(1+t^2)\,dt$



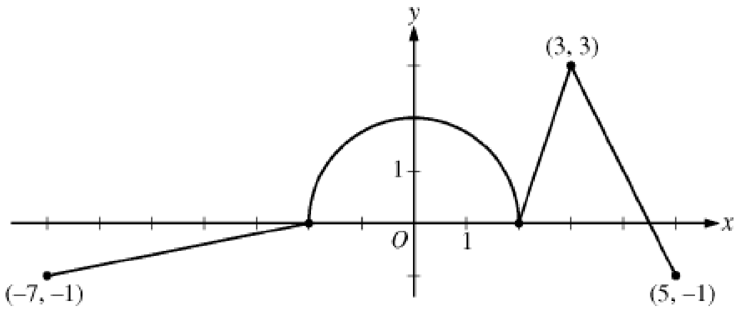
Graph of f

9. The continuous function f is defined on the interval $-4\le x\le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x)=2x+\int_0^x f(t)\,dt$. Find each value below.

a) $g(-4)$ b) $g(-3)$ c) $g(0)$ d) $g(1.5)$

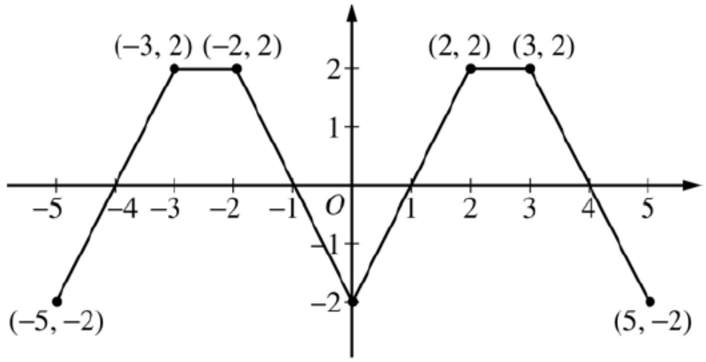
e) $g'(-3)$ f) $g'(0)$ g) $g'(1.5)$ h) $g''(-3)$

i) $g''(0)$ j) $g''(1)$



Graph of g'

10. The function g is defined and differentiable on the closed interval $[-7,5]$ and satisfies $g(0)=5$. The graph of $y=g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure at the right. Find $g(3)$ and $g(-2)$.



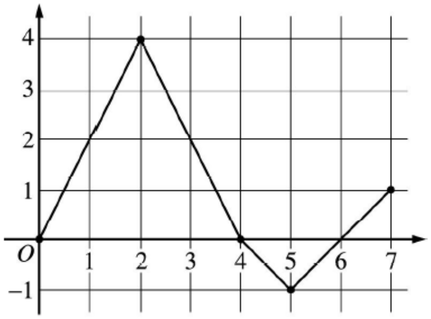
Graph of f

11. The graph of the function f shown at right consists of six line segments. Let g be the function given by $g(x)=\int_0^x f(t)\,dt$. Complete the table below.

x	$g(x)$	$g'(x)$	$g''(x)$
-4			
-1			
4			

AP Calculus BC - Worksheet 52 **The Accumulation Function and Graphical Analysis**

2003 (Form B) #3 (Calculator)



Graph of f

Let f be a function defined on the closed interval $[0, 7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

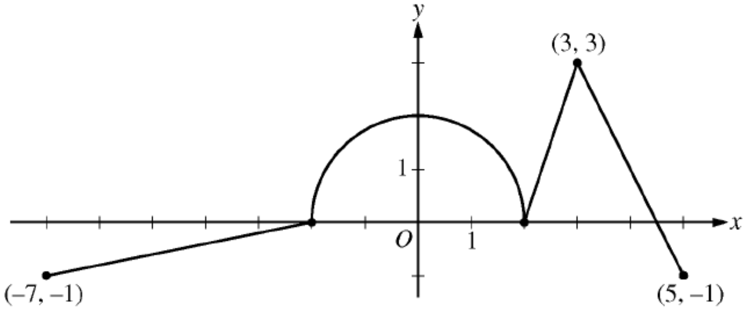
(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \leq x \leq 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b) ? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

2010 #5 (No Calculator)



Graph of g'

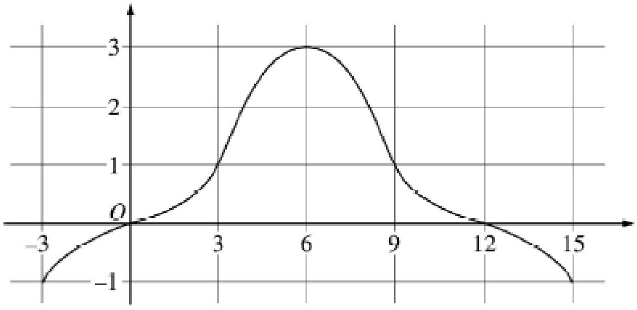
The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find $g(3)$ and $g(-2)$.

(b) Find the x-coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.

(c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x-coordinate of each critical point of h, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

2002 (Form B) #4 (No Calculator)



Graph of f

The graph of a differentiable function f on the closed interval $[-3, 15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.

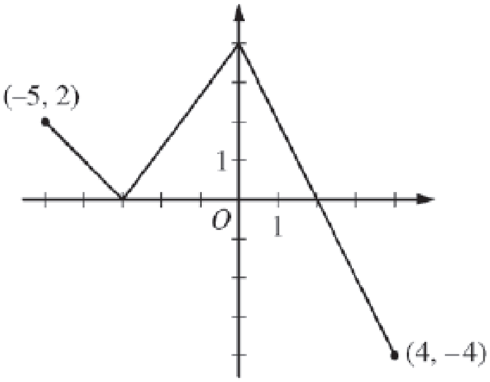
(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

2013 BC#3 (No Calculator)



Graph of f

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

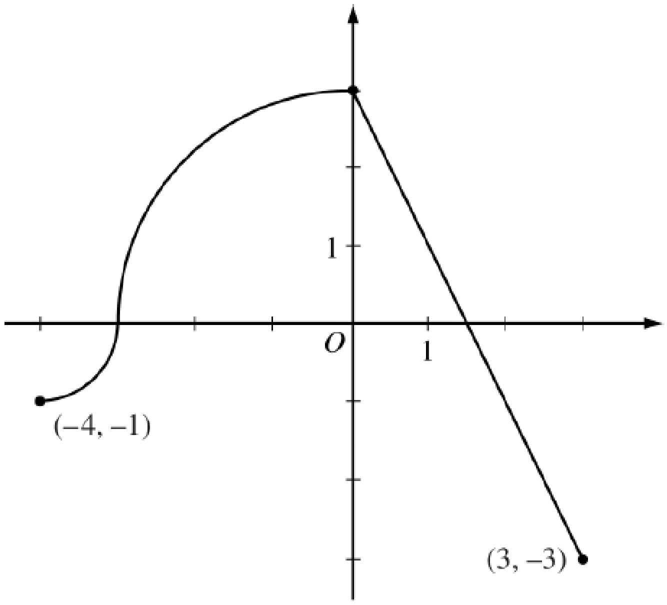
(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

2014 BC #4 (No Calculator)



Graph of f

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

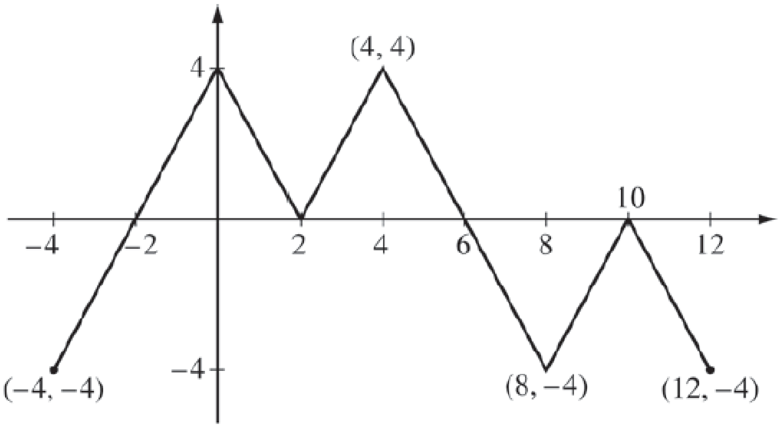
(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

2016 BC#3 – No Calculator



Graph of f

. The figure above shows the graph of the piecewise-linear function f. For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.

(d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

AP Calculus BC – Worksheet 53 Average Value of a Function

1 CALCULATOR PERMITTED
Let f be the function given by $f(x)=3\ln\left(2+x^2\right)\cos x$. What is the average value of *f* on the closed interval $2\le x\le 6$?

2 The average value of a function *f* over the interval $[-1,2]$ is -4, and the average value of *f* over the interval $[2,7]$ is 8. What is the average value of *f* over the interval $[-1,7]$.

3 CALCULATOR PERMITTED
Let f be the function given by $f(x)=\sin(2x)\cos(1+x)$. What is the average value of *f* on the closed interval $1\le x\le 3$?

4 The intensity of light at a distance *x* meters from a source is modeled by the function *R* given by $R(x)=\dfrac{k}{x^2}$, where *k* is a positive constant. Which of the following gives the average intensity of the light between 4 meters and 6 meters from the source?

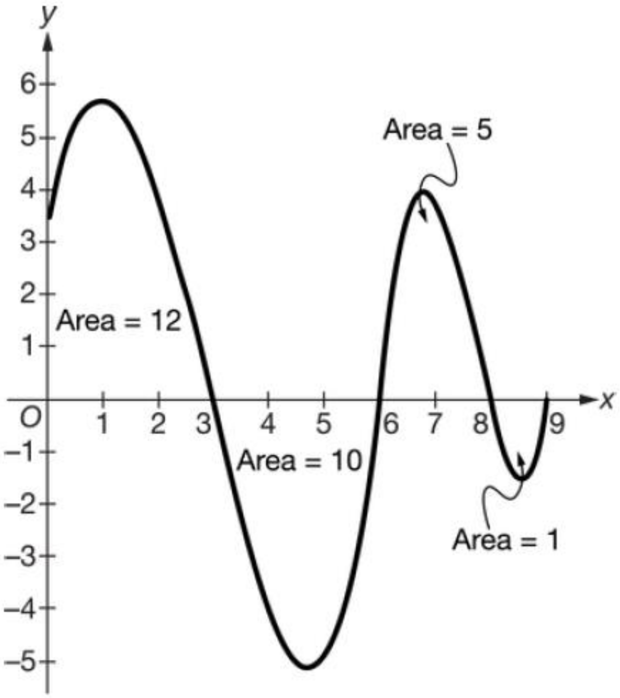
5

t (hours)	**0**	**$\frac{1}{2}$**	**1**	**$\frac{3}{2}$**	**2**
$R(t)$ (gallons per hour)	**16**	**12**	**7**	**3**	**0**

The rate at which oil leaks form a drum is modeled by the twice-differentiable function *R*, where $R(t)$ is measured in gallons per hour and *t* is measured in hours for $0\le x\le 2$. Values of $R(t)$ are given in the table above for selected values of *t*.

(a) Use the data in the table to find an approximation for $R'\left(\dfrac{3}{4}\right)$. Show the computations that lead to your answer. Indicate units of measure.
(b) Use a trapezoidal sum with four subintervals indicated by the data in the table to approximate $\int_0^2 R(t)\,dt$. Indicated units of measure.
(c) Use the data in the table to evaluate $\int_0^1 R'(t)\,dt$.
(d) The sum $\sum_{k=1}^{n} R\left(\dfrac{1}{2}+\dfrac{k}{n}\right)\dfrac{1}{n}$ is a right Riemann sum with *n* subintervals of equal length. The limit of this sum as *n* goes to infinity can be interpreted as a definite integral. Express the limit as a definite integral.

6



Graph of f'

The figure above shows the graph of f', the derivative of differentiable function *f*, on the closed interval $0 \le x \le 9$. The areas of the regions between the graph of f' and the *x*-axis are labeled in the figure. The function *f* is defined for all real numbers and satisfies $f(6) = 7$.

Let *g* be the function defined $g(x) = x^2 - 1$.

(a) Find the value of $\int_0^9 f'(x)dx$

(b) Given that $f(6) = 7$, write an expression for $f(x)$ that involves an integral. Use this expression to find the absolute minimum value of *f* and the absolute maximum value of *f* on the closed interval $0 \le x \le 9$. Justify your answers.

(c) Find $\int g(x)dx$.

(d) Find the value of $\int_2^3 xf'(g(x))dx$

7

Let *f* be the function defined by $f(x) = \dfrac{4}{x^2}$.

(a) Approximate the value of $\int_1^{12} f(x)dx$ using a left Riemann sum with the subintervals $[1,4],[4,8],$ and $[8,12]$.

(b) Find the value of $\int_1^{12} f(x)dx$.

(c) Find the value of $\int_1^{\infty} f(x)dx$.

(d) Find the value of $\int_1^{12} f(x) \cdot \ln x dx$. Show the work that leads to your answer.

AP Calculus BC – Worksheet 54 **Particle Motion (Definite Integrals)**

1) The table below gives the velocity of an object. Using the values given, estimate $\int_0^{15} |v(t)|\,dt$ using a trapezoidal approximation. Explain the meaning of $\int_0^{15} |v(t)|\,dt$.

t	0	2	5	6	8	12	15
$v(t)$	3	7	8	-6	12	-10	9

2)

The values of the coordinate s of a bug moving smoothly and continuously along a line for various values of $t \in [0,4]$ are given below.

t	0	0.5	1	1.5	2	2.7	3	3.6	4
$s(t)$	40.0	35.0	30.2	36.0	48.2	45.0	38.2	16.0	0.2

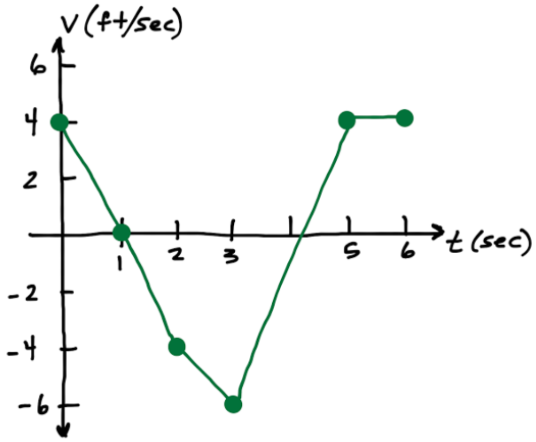
(a) What is the displacement of the bug during the given interval $t \in [0,4]$? Show the work that leads to your answer.

(b) What is the minimum number of times the bug changes directions for $t \in [0,4]$? Explain your reasoning.

(c) What is the bug's average velocity for $t \in [0,0.5]$? Show the work that leads to your answer.

(d) Estimate the bug's velocity at each of the following. Use proper notation (always), and show the work that leads to your answers.

(i) At $t = 0.5$ (ii) At $t = 2.7$ (iii) At $t = 3.5$



3)

(a) On what open intervals or at what time(s) $0 < t < 6$ is the particle at rest? Justify.

(b) On what open intervals $0 < t < 6$ is the particle moving to the right? Justify.

(c) On what open intervals or at what time(s) $0 < t < 6$ is the particle moving at its greatest speed? Greatest velocity?

(d) On what open intervals or at what time(s) $0 < t < 6$ is the particle's speed increasing? Decreasing? Justify.

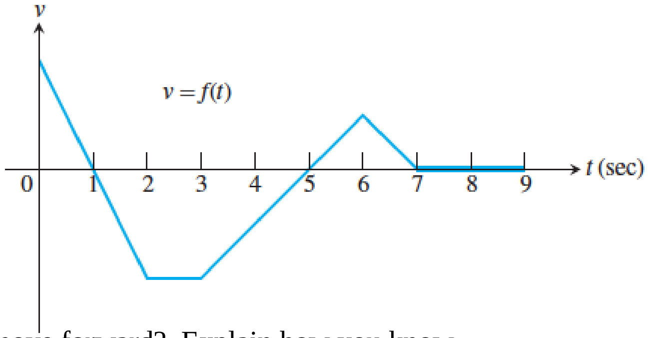
(e) What is the particle's acceleration at $t = 4.8$ second? Explain what this number means in terms of the particle's velocity.

(f) On what open intervals or at what time(s) $0 < t < 6$ is the acceleration of the particle the greatest?

(g) (is for "genius") What is the particle's displacement during the 2 seconds? Justify.

4)

The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line. The initial position of the particle is $s(0) = 4$



a) When does the particle move forward? Explain how you know.

b) When does the particle move backward? Explain how you know.

c) When does the particle change direction? Explain how you know.

d) When is the particle's acceleration positive? Negative?

e) What is the particle's displacement over the time interval $[0,9]$?

f) Determine the particle's position at $t = 4$.

g) Find the total distance travelled by the particle over the time interval $[0,9]$.

5)

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

AP Calculus BC – Worksheet 55 Integral as Net Change

Part I (Matching) - Read the problem and then match the integral expressions in the left column to their interpretations in the right column.

A pipeline company manufactures pipe that sells for $100 per meter. The cost of manufacturing a portion of the pipe varies with its distance from the beginning of the pipe. The company reports that the cost to produce a portion of the pipe that is *x* meters from the beginning of the pipe is $C(x)$ dollars per meter. (Note: Profit is defined as the difference between the amount of money received by the company for selling the pipe and the amount it costs to manufacture the pipe.)

1. $\int_0^{125} C(x)\,dx$	**A.** The difference in sales price, in dollars, between 100 metes of pipe and 50 meters of pipe. **B.** The cost, in dollars, of manufacturing 125 meters of pipe. **C.** The average profit, in dollars per meter, made on the sale of 125 meters of pipe. **D.** The difference in the cost, in dollars, of manufacturing 100 meters of pipe and 50 meters of pipe. **E.** The average rate of change, in dollars per meter per meter, in the cost per meter for 125 meters of pipe. **F.** The difference in profit, in dollars, between selling 100 meters of pipe and 50 meters of pipe. **G.** The average difference in the cost per meter, in dollars per meter, between manufacturing 100 meters of pipe and 50 meters of pipe.
2. $\frac{1}{125}\int_0^{125} \left[100 - C(x)\right] dx$	
3. $\frac{1}{125}\int_0^{125} \left[100 - C'(x)\right] dx$	
4. $\int_{50}^{100} C(x)\,dx$	
5. $\frac{1}{50}\int_{50}^{100} C(x)\,dx$	
6. $\int_{50}^{100} 100\,dx$	
7. $\int_{50}^{100} \left[100 - C(x)\right] dx$	

Part II - Please answer all questions in the context of the specific problems. Read each question carefully as not all parts involve integration.

1. At the beginning of an eight-hour shift, there are 55 widgets waiting to be shipped. New widgets are produced at the rate of $W(t)$, measured in widgets per hour. Widgets are shipped out of the factory at a rate of $S(t)$, also measured in widgets per hour. Write a definite integral to find each of the following.
   (a) The total number of widgets produced during the eight-hour shift.
   (b) The change in the number of widgets waiting to be shipped between the end of the second hour and the beginning of the fourth hour of the shift.
   (c) The number of widgets that are waiting to be shipped at the end of the eight-hour shift.
   (d) The total number of widgets shipped during the second half of the shift.

2. (NO CALCULATOR) The number of gallons of water in a tank *t* minutes after the tank has started to drain is $Q(t)=200(3-t)^2$.

   (a) How fast is the water running out at the end of 10 minutes? Indicate units of measure.

   (b) What is the average rate at which the water flows out in the time interval $[0,10]$?

3. (CALCULATOR PERMITTED) A spherical tank contains 81.637 gallons of water at time $t=0$ minutes. For the next 6 minutes, water flows out of the tank at a rate of $9\sin\left(\sqrt{t+1}\right)$ gallons per minute.

   (a) Find the rate at which water flows out of the tank when $t=4$ minutes. Indicate units of measure.

   (b) Write a general function $W(t)$ that can be used to determine the amount of water in the tank at any time, *t*.

   (c) How many gallons of water are in the tank at the end of 6 minutes?

4. (CALCULATOR PERMITTED) The rate at which water is sprayed on a field of vegetables is given by $R(t)=2\sqrt{t+5t^3}$, where *t* is in minutes and $R(t)$ is in gallons per minute.

   (a) Write a general function $W(t)$ that can be used to determine the amount of water sprayed on the field at any time, *t*.

   (b) How much water has been sprayed on the field during the first 10 minutes?

5. (CALCULATOR PERMITTED) The tide removes sand from Sandy Point Beach at a rate modeled by the function *R* given by $R(t)=2+5\sin\left(\dfrac{4\pi t}{25}\right)$. How much sand will have been removed by the tide after 6 hours?

6. (CALCULATOR PERMITTED) The rate at which raw sewage enters a treatment tank is given by $E(t)=850+715\cos\left(\dfrac{\pi t^2}{9}\right)$ gallons per hour for $0\le t\le 4$ hours. Treated sewage is removed from the tank at a constant rate of 645 gallons per hour. The treatment tank is empty at time $t=0$.

   (a) At what rate is the sewage entering the tank when $t=3$ hours?

   (b) How many gallons of sewage enter the treatment tank during the time interval $0\le t\le 4$? Round your answer to the nearest gallon.

   (c) How many gallons of sewage are in the tank at $t=3$ hours?

AP Calculus BC – Worksheet 56 — Rates – AP Free Response Questions

2016 – BC1 (Calculator Permitted)

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

2015 – BC1 (Calculator Permitted)

The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

2014 – BC1 (Calculator Permitted)

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

2013 – BC1 (Calculator Permitted)

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

AP Calculus BC – Worksheet 57 Areas in a Plane

I. For each of the following, find the points of intersection of the graphs of the two functions, and use a definite integral to find the area of the region bounded by the given functions.

1-6 No Calculator
7-11 Calculator Permitted

1. $y = x^2$ $y = 4x$ **2.** $y = e^x$ $y = \sqrt{x}$ $x = 0$ $x = 1$

3. $y = 4 - x^2$ $y = -4$ **4.** $y = x^3$ $y = x^2$

5. $x = y^2$ $x - y = 2$ **6.** $x = 4y - y^3$ $x = 0$

7. $x + y = 3$ $y + x^2 = 3$ **8.** $y = \sqrt{x}$ $y = -x$ $x = 1$ $x = 4$

9. $y = \cos x$ $y = x^2 + 10x + 16$ **10.** $y = \sin 4x$ $y = 1 + \cos\left(\frac{x}{3}\right)$ $x = 0$ $x = \pi$

11. $y = x^3 - 4x + 2$ $y = 2$

12. Let *R* be the region bounded by the graph of $f(x) = \left|x^2 - 6x + 5\right|$ from *x = 0* to *x = 7*. Set up a sum of integrals that do not contain an absolute value that can be used to find the area of *R*. Do not solve the problems.

AP Calculus BC - Worksheet 58 **Volumes with Known Cross Sections**

1. The base of a solid in the *xy*-plane is a right triangle bounded by the axes and $y = -x + 2$. Cross sections of the solid perpendicular to the *x*-axis are squares. Find the volume.

2. (Calculator Permitted) The base of a solid *S* is the region enclosed by the graph of $y = \ln x$, the vertical line $x = e$, and the *x*-axis. If the cross sections of *S* perpendicular to the *x*-axis are squares, which of the following gives the best approximation of the volume of *S*?
(A) 0.718 (B) 1.718 (C) 2.718 (D) 3.171 (E) 7.388

3. (Calculator Permitted) The base of a solid is the region in the first quadrant bounded by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$ and the *y*-axis. For this solid, each cross section perpendicular to the *x*-axis is a square. Find the volume of the solid.

4. (Calculator Permitted) The base of a solid is the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$, $y = e^{-3x}$ and the vertical line $x = 1$. For this solid each cross section perpendicular to the *x*-axis is a rectangle whose height is 5 times its length of its base. Find the volume of the solid.

5. (Calculator Permitted) The base of a solid is the region in the first quadrant bounded by the *x*-axis, the graph of $y = \sin^{-1} x$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the *y*-axis is a semicircle. What is the volume?
(A) 0.356 (B) 0.279 (C) 0.139 (D) 1.571 (E) 0.571

6. (Calculator Permitted) The base of a solid is the region bounded by the curve $y = 2 + \sin x$, the *x*-axis, $x = 0$, and $x = \frac{3\pi}{2}$. Find the volume of the solids whose cross sections perpendicular to the *x*-axis are the following:
   a) squares
   b) rectangles whose height is 3 times the base
   c) equilateral triangles
   d) isosceles right triangles with leg on the base
   e) isosceles right triangles with hypotenuse on the base
   f) semi-circles
   g) quarter-circles

Answers:

1. $\frac{8}{3}$	2. A	3. 0.461	4. 1.554
5. C	6. a) 25.20575 b) 75.61725 c) 10.91441 d) 12.60287 e) 6.301437 f) 9.898275 g) 19.79655		

AP Calculus BC – Worksheet 59

Volumes of Solids of Rotation

1

(Calculator Permitted) Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{3/2}$, the x-axis, and the y-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the x-axis?

(A) 60.3 (B) 115.2 (C) 225.4 (D) 319.7 (E) 361.9

2

Let R be the region enclosed by the graph of $y = x^2$, the line $x = 4$, and the x-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the y-axis.

(A) 64π (B) 128π (C) 256π (D) 360 (E) 512

3

Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^x$, and $x = 1$. Which of the following gives the volume of the solid generated when R is revolved about the x-axis?

(A) $\int_0^1 \left(e^x - e^{-x}\right) dx$ (B) $\int_0^1 \left(e^{2x} - e^{-2x}\right) dx$ (C) $\int_0^1 \left(e^x - e^{-x}\right)^2 dx$

(D) $\pi \int_0^1 \left(e^{2x} - e^{-2x}\right) dx$ (E) $\pi \int_0^1 \left(e^x - e^{-x}\right)^2 dx$

4

(Calculator Permitted) Let R be the region bounded by the curves $y = x^2 + 1$ and $y = x$ for $0 \le x \le 1$. Showing all integral set-ups, find the volume of the solid obtained by rotating the region R about the

a) x-axis
b) line $y = -1$
c) line $y = 3$
d) y-axis

Answers

1) E	2) B
3) D	4) a) 4.817 b) 10.053 c) 10.890 d) 2.617

AP Calculus BC – Worksheet 60 **Volume**

1

Let R be the region in the first quadrant bounded by the graph of $y=3x-x^2$ and the x-axis. A solid is generated when R is revolved about the vertical line $x=-1$. Set up, but do not evaluate, the definite integral that gives the volume of this solid.

(A) $\int_0^3 2\pi(x+1)\left(3x-x^2\right)dx$ (B) $\int_{-1}^3 2\pi(x+1)\left(3x-x^2\right)dx$ (C) $\int_0^3 2\pi(x)\left(3x-x^2\right)dx$

(D) $\int_0^3 2\pi\left(3x-x^2\right)^2 dx$ (E) $\int_0^3 \left(3x-x^2\right)dx$

2

(Calculator Permitted) Let R be the region bounded by the graphs of $y=\sqrt{x}$, $y=e^{-x}$, and the y-axis.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the line $y=-1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle whose diameter runs from the graph of $y=\sqrt{x}$ to the graph of $y=e^{-x}$. Find the volume of this solid.

3

(Calculator Permitted)

Find the volume of the solid formed when the R enclosed by the curves $y=x$ and $y=x^2$ is rotated about the following axes:

a) the x-axis
b) the line $y=-1$
c) the line $y=7$
d) the y-axis
e) the line $x=-1$
f) the line $x=17$.

Answers:

1) A	2) a) 0.161 b) 1.630 c) 0.034	3) a) 0.418 b) 1.466 c) 6.911 d) 0.523 e) 1.570 f) 17.278

AP Calculus BC – Worksheet 61 **Arc Length**

1

('88 BC) The length of the curve $y = x^3$ from $x = 0$ to $x = 2$ is given by

(A) $\int_0^2 \sqrt{1+x^6}\,dx$ (B) $\int_0^2 \sqrt{1+3x^2}\,dx$ (C) $\pi\int_0^2 \sqrt{1+9x^4}\,dx$

(D) $2\pi\int_0^2 \sqrt{1+9x^4}\,dx$ (E) $\int_0^2 \sqrt{1+9x^4}\,dx$

2

('03 BC) The length of a curve from $x = 1$ to $x = 4$ is given by $\int_1^4 \sqrt{1+9x^4}\,dx$. If the curve contains the point $(1, 6)$, which of the following could be an equation for this curve?

(A) $y = 3 + 3x^2$ (B) $y = 5 + x^3$ (C) $y = 6 + x^3$

(D) $y = 6 - x^3$ (E) $y = \frac{16}{5} + x + \frac{9}{5}x^5$

3

(Calculator Permitted) Which of the following gives the best approximation of the length of the arc of $y = \cos(2x)$ from $x = 0$ to $x = \frac{\pi}{4}$?

(A) 0.785 (B) 0.955 (C) 1.0 (D) 1.318 (E) 1.977

4

Which of the following gives the length of the graph of $x = y^3$ from $y = -2$ to $y = 2$?

(A) $\int_{-2}^{2} \left(1+y^6\right)dy$ (B) $\int_{-2}^{2} \sqrt{1+y^6}\,dy$ (C) $\int_{-2}^{2} \sqrt{1+9y^4}\,dy$ (D) $\int_{-2}^{2} \sqrt{1+x^2}\,dx$ (E) $\int_{-2}^{2} \sqrt{1+x^4}\,dx$

5

Find the length of the curve described by $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$.

(A) $\frac{26}{3}$ (B) $\frac{52}{3}$ (C) $\frac{512\sqrt{2}}{15}$ (D) $\frac{512\sqrt{2}}{15} + 8$ (E) 96

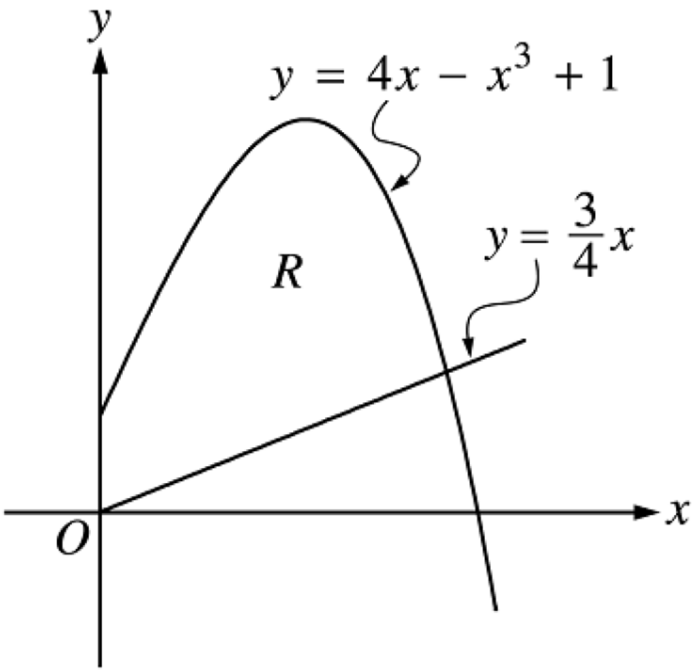
6

Which of the following expressions should be used to find the length of the curve $y = x^{2/3}$ from $x = -1$ to $x = 1$?

(A) $2\int_0^1 \sqrt{1+\frac{9}{4}y}\,dy$ (B) $\int_{-1}^{1} \sqrt{1+\frac{9}{4}y}\,dy$ (C) $\int_0^1 \sqrt{1+y^3}\,dy$ (D) $\int_0^1 \sqrt{1+y^6}\,dy$ (E) $\int_0^1 \sqrt{1+y^{9/4}}\,dy$

7

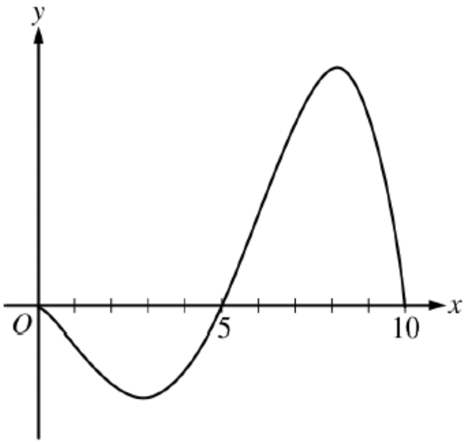
(AP BC 2002B-3) (Calculator Permitted) Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = 4x - x^3 + 1$ and $y = \frac{3}{4}x$.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Write an expression involving one or more integrals that gives the perimeter of R. Do not evaluate.



Graph of *f*

4. The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.

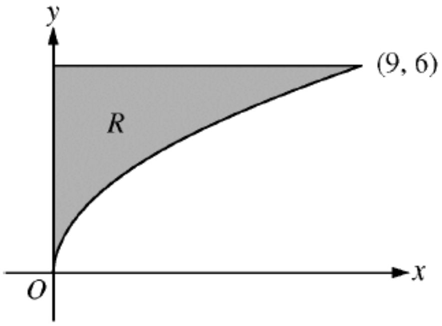
   (a) Find the average value of f on the interval $0 \le x \le 5$.

   (b) Evaluate $\int_0^{10} (3f(x) + 2)\,dx$. Show the computations that lead to your answer.

(c) Let $g(x) = \int_5^x f(t)\,dt$. On what intervals, if any, is the graph of g both concave up and decreasing? Explain your reasoning.

(d) The function h is defined by $h(x) = 2f\left(\frac{x}{2}\right)$. The derivative of h is $h'(x) = f'\left(\frac{x}{2}\right)$. Find the arc length of the graph of $y = h(x)$ from $x = 0$ to $x = 20$.

No Calculator on this problem



1. Let *R* be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the $y-\text{axis}$, as shown in the figure above.

   a) Find the area of *R*.

   b) Write, but do not solve, an integral expression that can be used to find the perimeter of *R*.

   c) Region *R* forms the base of solid. Write, but do not evaluate, an integral expression that can be used to find the volume of this solid if cross-sections taken perpendicular to the *x*-axis are semicircles.

   d) Region *R* is rotated around the line $y = 6$ to form a solid. Write, but do not solve, an integral expression that can be used to find the volume of the solid.

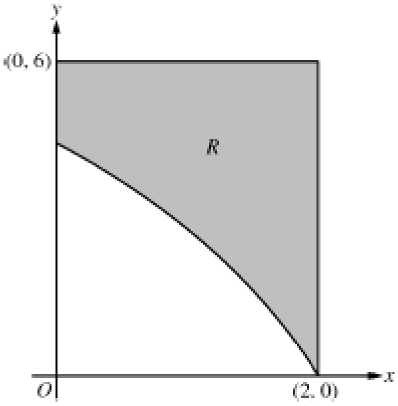
   e) Region *R* is rotated around the *x*-axis to form a solid. Write, but do not solve, an integral expression that can be used to find the volume of the solid.

   f) Region *R* is rotated around the *y*-axis to form a solid. Write, but do not solve, an integral expression that can be used to find the volume of the solid.

   g) Region *R* is rotated around the line $x = 9$ to form a solid. Write, but do not solve, an integral expression that can be used to find the volume of the solid.

2. Approximate the area under the graph of $f(x) = -x^2 + 5$ on the interval $[0, 2]$ using 4 subintervals:
   a) LRAM
   b) RRAM
   c) MRAM

A calculator may be used for these problems.



3. In the figure above, *R* is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

   a) Find the area of *R*.

   b) Find the perimeter of *R*.

   c) The region *R* is the base of a solid. For this solid, each cross section perpendicular to the *x*-axis is a square. Find the volume of the solid.

   d) The region *R* is the base of a solid. For this solid, each cross section perpendicular to the *y*-axis is a quarter-circle. Find the volume of the solid.

   e) Find the volume of the solid generated when *R* is revolved about the horizontal line $y = 6$.

   f) Find the volume of the solid generated when *R* is revolved about the horizontal line $y = 8$.

   g) Find the volume of the solid generated when *R* is revolved about the vertical line $x = 2$.

   h) Find the volume of the solid generated when *R* is revolved about the vertical line $x = -11$.

4. Selected values of $f(x)$ are given in the table below. Approximate the value of $\int_0^9 f(x)\,dx$ using 4 subintervals:

x	0	2	3	7	0
$f(x)$	3	6	7	6	8

   a) LRAM
   b) MRAM